

Design of Binary Network Codes for Multi-user Multi-way Relay Networks

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Abstract

We study multi-user multi-way relay networks where N user nodes exchange their information through a single relay node. We use network coding in the relay to increase the throughput. Due to the limitation of complexity, we only consider the binary multi-user network coding (BMNC) in the relay. We study BMNC matrix (in $GF(2)$) and propose several design criteria on the BMNC matrix to improve the symbol error probability (SEP) performance. Closed-form expressions of the SEP of the system are provided. Moreover, an upper bound of the SEP is also proposed to provide further insights on system performance. Then BMNC matrices are designed to minimize the error probabilities.

Index Terms

N -way relay, binary network coding, symbol error probability.

I. INTRODUCTION

Network coding (NC) is considered as a potentially powerful tool for efficient information transmission in wireless networks, where data flows coming from multiple sources or to different

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sinks are combined to increase throughput, reduce delay, or enhance robustness [1]–[3]. Consider a two-way wireless system where two source nodes communicate with each other through the aid of one relay node [4]–[8]. With network coding, each of the two transceivers employs one time slot to transmit a packet to the relay in a conventional time-division multiple access (TDMA) scheme. Next, the relay takes the exclusive-or of these two packets and broadcasts the result during the third time slot. Armed with the packet it sent to the relay, each of the transceivers can then recover the data originating at the other relay, with the network having only used three slots rather than the traditional four. In what follows, we shall extend NC to the multi-user, multi-hop, multi-relay and multi-radio wireless ad hoc networks, which are introduced in [9]–[13].

As previous related work, the sink bit error probability (BEP) for the coded network with memoryless and independent channels is investigated in [14]. The alphabet size of the code is $\text{GF}(2^m)$. In [15], finite-field network coding (FFNC) is designed for multiple-user multiple-relay (MUMR) wireless networks with quasi-static fading channels. For high rate regions, FFNC has significantly better performance than superposition coding. In [16], using code division multiple access (CDMA) of an interference limited system, a jointly demodulate-and-XOR forward (JD-XOR-F) relaying scheme is proposed, where all users transmit to the relay simultaneously followed by the relay broadcasting an estimate of the XORed symbol for each user pair. The problem of joint resource allocation for OFDMA assisted two-way relay system is studied in [17] and the objective function is to maximize the sum-rate through joint subcarrier allocation, subcarrier pairing, and power allocation, under the individual power constraints at each transmitting node. Several beamforming schemes are proposed in [18] for the scenario where multiple pairs of users exchange information within pair, with the help of a dedicated multi-antenna relay. A cooperation protocol based on complex-field wireless network coding is developed in a network with N sources and one destination [19]. To deal with decoding errors at sources, selective- and adaptive-forwarding protocols are also developed at no loss of diversity gain. For the multiple-access relay network, the capacity approaching behavior of the joint network LDPC code is analyzed in [20], [21].

Above literatures focus on the information exchange of multiple pairs of users with or without

the assistance of one relay node. For more general cases, in a practical network, there are multiple relays or multiple hops. In [22], new approaches to LDPC code design for a multi-source single-relay FDMA system are explored, under the assumption of uniform phase-fading Gaussian channels. In [23], a binary field NC design over a multiple-source multiple-relay wireless network over slow-fading channels is studied. In [24], a novel scheme of multi-channel/interface network coding is proposed, which is based on the combination of a new concept of coded-overhearing and coding-aware channel assignment. In [25], the power allocation policies are investigated across the relays for automatic gain control (AGC)-based amplify-and-forward (AF) distributed space-time code (DSTC) systems in the two-way relay networks. In [26], with a new flow-based characterization of pairwise intersession network coding, an optimal joint coding, scheduling, and rate-control scheme can be devised and implemented using only the binary XOR operation. In [27], a novel concept of wireless network cocast (WNC) [28] is considered and its associated space-time network codes (STNCs) are proposed to achieve the foretold objectives. However, CDMA-like, FDMA-like and TDMA-like techniques are proposed in [27], [28], where each symbol is assigned a complex-valued signature waveform, the dedicated carrier and the symbol duration. In [29], several interesting properties of network coding matrices are discussed in a network where N users have independent information to send to a common base station.

In [30], it has been shown that for the N -way single-channel relay network, it takes at least $(2N - 1)$ time slots for the linear NC scheme without opportunistic listening to perform a round of the N -way relay, where there are N , with $N \geq 2$, end nodes exchanging their information with the assistance of one N -way relay with single antenna. However, [30] focus on a general linear programming framework for solving the throughput optimization problems and a joint link scheduling, channel assignment, and routing algorithm for the wireless NC schemes to closely approximate the optimal solutions. The detailed linear NC for N -way single-channel relay networks, such as how N information packets are encoded into $N - 1$ pronumerals, is not investigated in [30].

In this paper, we take a step further to investigate the efficient linear NC for N -way single-channel relay network, which is also discussed in [30]. As shown in [23], in the case that

the network size (i.e., the number of the sources and the number of the relays) and the frame length (i.e., the number of symbols or Galois field elements in a frame) are large, we need to choose a large size of the Galois fields. Therefore, the encoding complexity of the GF(q) codes will significantly increase. Since binary network coding is of low complexity, binary multi-user network coding (BMNC) is considered here to increase throughput. Several design criteria that the BMNC matrix should follow to increase the system performance are provided. Moreover, the effects of the noise and the BMNC matrix are studied, based on which the symbol error probability (SER) of the system is provided. To improve the system performance further, BMNC matrices are designed for arbitrary number of users, which minimize the bound of SEP.

The paper is organized as follows. In Section II, the system model is introduced. In Section III, BMNC decoding process and a design criterion on the BMNC matrix are presented. Performance analysis is shown in Section IV, which includes BMNC matrix analysis, closed-form expressions of the SEP and throughput of the system, the tight upper bound of SEP of the system and the designed BMNC encoding matrix. In Section V, the optimality of the matrix given in Section IV-D is discussed. Simulation results are presented in Section VI and the conclusions are given in Section VII.

Some notations are listed as follows. Symbol $(\mathbf{A})_{i,j}$ presents the $(i,j)^{\text{th}}$ element of matrix \mathbf{A} . Symbols \oplus , \sum^{\oplus} denote the addition and the summation in GF(2), respectively. Symbol \circ presents element-wise product and $(\mathbf{A} \circ \mathbf{B})_{i,j} = (\mathbf{A})_{i,j} (\mathbf{B})_{i,j}$. Symbol \prod° presents the element-wise product of multiple matrices or vectors.

II. SYSTEM MODEL

Consider a wireless network with N user nodes U_i , $i = 1, \dots, N$, and one relay node R , as shown in Fig. 1. Each node has only one antenna, which can be used for both transmission and reception. For practical services such as video conference in which each user may want to have a discussion with the other users, the N users need the information of other users and they exchange information with the assistance of the relay R . Without loss of generality, in one time slot, the exchanged information bit of U_i can be denoted by x_i , $1 \leq i \leq N$. Whereas, in practice,

the user nodes and the relay will transmit the information in packets that contains a large number of symbols. The user nodes will collect all the transmitted packets and then jointly detect them. We assume that the direct links between the users are not available. All communications must be through the relay.

Take U_i for example, it needs the information from the other $N - 1$ users, while the other $N - 1$ users need the information of U_i . In the traditional scheme, considering the time division transmission schemes, the traditional scheme needs $2N$ time slots to finish the information exchange, where N time slots are used for the relay to receive the N information bits of the N user nodes and the other N time slots are used for the user nodes to receive the N information bits from the relay.

In order to improve the system performance, we propose a BMNC scheme, in which only $2N - 1$ time slots are used. In this scheme, the transmission can be divided into two consecutive phases. 1) In the source transmission phase, each user node sends its own information to the relay node, which takes N time slots. The relay receives and then detects the N information bits from the N users. 2) In the relay transmission phase, the relay linearly combines the detected information bits, and then broadcasts the combined information bits to all the users. Since each user knows its own information, only the information bits of other $N - 1$ users are needed. Thus, at least $N - 1$ information bits should be broadcasted from the relay to all the users. Finally, the BMNC scheme takes $2N - 1$ time slots to achieve the information exchange.

In the source transmission phase, the received symbols at the relay node are

$$\mathbf{y}_0 = \mathbf{H}_0 M(\mathbf{x}) + \mathbf{N}_0, \quad (1)$$

where $\mathbf{y}_0 = \begin{bmatrix} y_{0,1} & y_{0,2} & \dots & y_{0,N} \end{bmatrix}^T$ denotes the received signals at the relay, $\mathbf{H}_0 = \text{diag}\{h_{0,1}, h_{0,2}, \dots, h_{0,N}\}$ denotes the fading coefficients, $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^T$ denotes the bits of the users, $M(\cdot)$ denotes the modulation transformation and $M(\mathbf{x})$ denotes the transmitted symbols of the users, and $\mathbf{N}_0 = \begin{bmatrix} n_{0,1} & n_{0,2} & \dots & n_{0,N} \end{bmatrix}^T$ denotes the additive white Gaussian noise (AWGN) with zero mean.

Then the relay detects the received information and obtains an estimation of the source bits $\tilde{\mathbf{x}} = [\tilde{x}_1 \ \tilde{x}_2 \ \dots \ \tilde{x}_N]^T$. Then linearly network coding is proposed to combine N information bits into $N - 1$ bits, which can be shown as follows

$$(\mathbf{F}\tilde{\mathbf{x}}) \bmod (2) = \mathbf{r}, \quad (2)$$

where \mathbf{F} is the network encoding matrix in $\text{GF}(2)$, vector $\mathbf{r} = [r_1 \ r_2 \ \dots \ r_{N-1}]^T$ denotes the $N - 1$ information bits that the relay will broadcast. Note that r_i is the information bit to be transmitted in time slot i , $i \in [1, N - 1]$. The network encoding matrix \mathbf{F} can be described as

$$\begin{aligned} \mathbf{F} &= \begin{bmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,N} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N-1,1} & f_{N-1,2} & \cdots & f_{N-1,N} \end{bmatrix}_{(N-1) \times N} \\ &= [\mathbf{f}_1 \ \mathbf{f}_2 \ \cdots \ \mathbf{f}_N]_{(N-1) \times N}, \end{aligned} \quad (3)$$

where $f_{j,i}$ is one element of the network encoding matrix in $\text{GF}(2)$ for $j \in [1, N - 1]$, $i \in [1, N]$, which is related to U_i and the j th symbol that the relay sends. Vector $\mathbf{f}_i = [f_{1,i} \ f_{2,i} \ \cdots \ f_{N-1,i}]^T$ is the i th column vector of \mathbf{F} , which denotes the relationship between U_i and the $N - 1$ symbols that the relay sends.

The symbol $\text{GF}(2)$ is referred to the Galois field of two elements [31]. In our work, it consists of 0 and 1. Over $\text{GF}(2)$, many well-known but important properties of traditional number systems, such as real number, rational number etc., are retained: addition has an identity element and an inverse for every element; multiplication has an identity element “1” and an inverse for every element but “0”; addition and multiplication are commutative and associative; multiplication is distributive over addition [31].

In the relay transmission phase, the information that U_i receives is

$$\mathbf{y}_i = \mathbf{H}_i M(\mathbf{r}) + \mathbf{N}_i, \quad (4)$$

where $\mathbf{y}_i = \begin{bmatrix} y_{i,1} & y_{i,2} & \dots & y_{i,N-1} \end{bmatrix}^T$ denotes the signals that U_i receives, $\mathbf{H}_i = \text{diag}\{h_{i,1}, h_{i,2}, \dots, h_{i,N-1}\}$ denotes the fading coefficients, and $\mathbf{N}_i = \begin{bmatrix} n_{i,1} & n_{i,2} & \dots & n_{i,N-1} \end{bmatrix}^T$ denotes the AWGN with zero mean. Then the relay detects the received information \mathbf{y}_i and obtains $\tilde{\mathbf{r}}_i = \begin{bmatrix} \tilde{r}_{i,1} & \tilde{r}_{i,2} & \dots & \tilde{r}_{i,N-1} \end{bmatrix}^T$. Finally U_i decodes the information of other users through $\tilde{\mathbf{r}}_i, \mathbf{F}, x_i$.

III. BMNC DECODING PROCESS

As discussed above, the relay needs to broadcast at least $N - 1$ coded bits. However, arbitrary encoding may cause some users can not decode the source information bits even though there is no noise in the system. Thus, the network coding process should be designed carefully.

Clearly, U_i only knows its own information x_i , the $N - 1$ bits $\tilde{\mathbf{r}}_i$ that it detects from the received information and the network coding matrix \mathbf{F} . Then we investigate the relationship between $x_i, \tilde{\mathbf{r}}_i$ and \mathbf{F} . From (2), we have

$$\mathbf{r} = \begin{bmatrix} \sum_{k=1}^N \oplus f_{1,k} \tilde{x}_k \\ \sum_{k=1}^N \oplus f_{2,k} \tilde{x}_k \\ \vdots \\ \sum_{k=1}^N \oplus f_{N-1,k} \tilde{x}_k \end{bmatrix}_{(N-1) \times 1}. \quad (5)$$

Separating the information of U_i and other users, the above equation can be rewritten as

$$\begin{aligned} \mathbf{r} &= \begin{bmatrix} \sum_{k=1, k \neq i}^N \oplus f_{1,k} \tilde{x}_k \oplus f_{1,i} \tilde{x}_i \\ \sum_{k=1, k \neq i}^N \oplus f_{2,k} \tilde{x}_k \oplus f_{2,i} \tilde{x}_i \\ \vdots \\ \sum_{k=1, k \neq i}^N \oplus f_{N-1,k} \tilde{x}_k \oplus f_{N-1,i} \tilde{x}_i \end{bmatrix}_{(N-1) \times 1} \\ &= \{(\mathbf{F}_i \tilde{\mathbf{x}}_i) \bmod (2)\} \oplus \{(\mathbf{f}_i \tilde{x}_i) \bmod (2)\}, \end{aligned} \quad (6)$$

where $\mathbf{F}_i = \begin{bmatrix} \mathbf{f}_1 & \dots & \mathbf{f}_{i-1} & \mathbf{f}_{i+1} & \dots & \mathbf{f}_N \end{bmatrix}$ is the network sub-encoding matrix of U_i , $\tilde{\mathbf{x}}_i =$

$$\begin{bmatrix} \tilde{x}_1 & \cdots & \tilde{x}_{i-1} & \tilde{x}_{i+1} & \cdots & \tilde{x}_N \end{bmatrix}^T.$$

We denote $\hat{\mathbf{x}}_i = \begin{bmatrix} \hat{x}_{i,1} & \cdots & \hat{x}_{i,i-1} & \hat{x}_{i,i+1} & \cdots & \hat{x}_{i,N} \end{bmatrix}^T$ as the bits obtained by network decoding at U_i . In the BMNC decoding, based on (6), $\hat{\mathbf{x}}_i$ can be obtained through

$$\tilde{\mathbf{r}}_i = \{(\mathbf{F}_i \hat{\mathbf{x}}_i) \bmod (2)\} \oplus \{(\mathbf{f}_i x_i) \bmod (2)\}. \quad (7)$$

Adding $\{(\mathbf{f}_i x_i) \bmod (2)\}$ on both sides of the above equation, (7) can be rewritten as

$$(\mathbf{F}_i \hat{\mathbf{x}}_i) \bmod (2) = \tilde{\mathbf{r}}_i \oplus \{(\mathbf{f}_i x_i) \bmod (2)\}. \quad (8)$$

If the matrix \mathbf{F}_i is not full rank, the column vector $\{(\mathbf{f}_i x_i) \bmod (2)\}$ does not have $N - 1$ independent elements so that U_i can not obtain all the information bits of other $N - 1$ users. Thus \mathbf{F}_i should be full rank, then the inverse matrix \mathbf{F}_i^{-1} exists. Multiplying \mathbf{F}_i^{-1} on the two sides of (8), we have

$$\hat{\mathbf{x}}_i = \{\mathbf{F}_i^{-1} \{\tilde{\mathbf{r}}_i \oplus \{(\mathbf{f}_i x_i) \bmod (2)\}\}\} \bmod (2). \quad (9)$$

It can be seen that when \mathbf{F}_i is full rank, U_i can decode the information through (9). If \mathbf{F}_i is not full rank, U_i can not obtain all the information bits of other users. Thus, \mathbf{F}_i should be full rank for $i \in [1, N]$ to ensure that all the users can acquire the information of the other users.

Then we propose the following design criterion on the BMNC matrix to achieve the information exchange.

Theorem 1: For U_i , if \mathbf{F}_i is full rank, then

$$\mathbf{f}_i = \sum_{k=1, k \neq i}^N \oplus \mathbf{f}_k \quad (10)$$

is the necessary and sufficient condition that \mathbf{F}_j is full rank for $j \in [1, N]$.

Proof: See Appendix A. ■

Using Theorem 1, the network coding matrix \mathbf{F} can be easily designed through one full rank matrix in GF(2). Moreover, in the following, Theorem 1 is used for performance analysis.

IV. PERFORMANCE ANALYSIS

Above, network encoding and decoding protocols are investigated. From (9), it is evident that different \mathbf{F} results in different system performance. Thus, Theorem 1 is not sufficient for further performance analysis and improvement. In this section, the BMNC matrix and the error performance of the system will be analyzed.

A. Network coding matrix analysis

In what follows, we shall study how the network coding matrix affects error rates at U_i . First, we have the following result:

Theorem 2: Since U_i needs to obtain other $N - 1$ users' information bits, the error vectors that user i receives are

$$\begin{aligned}\hat{\mathbf{x}}_{e,i} &= \mathbf{x}_i \oplus \hat{\mathbf{x}}_i \\ &= (\mathbf{x}_i \oplus \tilde{\mathbf{x}}_i) \oplus ((x_i \oplus \tilde{x}_i) \mathbf{1}_{N-1 \times 1}) \oplus \{(\mathbf{F}_i^{-1}(\mathbf{r} \oplus \tilde{\mathbf{r}}_i)) \bmod (2)\},\end{aligned}\tag{11}$$

where $\mathbf{x}_i = \begin{bmatrix} x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_N \end{bmatrix}^T$ refers to the information that U_i wishes to obtain, $\mathbf{1}_{N-1 \times 1}$ is a column vector with $N - 1$ elements which are all 1.

Proof: See Appendix B. ■

B. Exact system performance

In this subsection, closed-form expressions of the SEP and throughput of the system will be derived.

The SEP of the system:

First, we shall investigate the addition in GF(2) and give the following lemma which will be used for our results later.

Lemma 3: For addition of Q numbers in GF(2), we have

$$\sum_{q=1}^Q \oplus a_q = \sum_{q=1}^Q (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \cdots < p_q \leq Q} \prod_{j=1}^q a_{p_j},\tag{12}$$

where $a_q \in \{0, 1\}$.

Proof: See Appendix C. ■

Since (11) is not convenient for SEP analysis, (11) should be transformed and the third item of (11) can be rewritten as

$$\begin{aligned} \{\mathbf{F}_i^{-1}(\mathbf{r} \oplus \tilde{\mathbf{r}}_i)\} \bmod (2) &= \left\{ \begin{bmatrix} \mathbf{G}_{i,1} & \mathbf{G}_{i,2} & \cdots & \mathbf{G}_{i,N-1} \end{bmatrix} (\mathbf{r} \oplus \tilde{\mathbf{r}}_i) \right\} \bmod (2) \\ &= \sum_{n=1}^{N-1} \oplus (\mathbf{G}_{i,n} \circ (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)), \end{aligned} \quad (13)$$

where vector $\mathbf{G}_{i,k}$ is the k th column vector of \mathbf{F}_i^{-1} .

Substituting (13) into (11), we have

$$\begin{aligned} \hat{\mathbf{x}}_{e,i} &= (\mathbf{x}_i \oplus \tilde{\mathbf{x}}_i) \oplus ((x_i \oplus \tilde{x}_i) \mathbf{1}_{N-1 \times 1}) \oplus \left(\sum_{n=1}^{N-1} \oplus (\mathbf{G}_{i,n} \circ (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)) \right) \\ &= \sum_{n=1}^{N+1} \oplus \mathbf{a}_{i,n}, \end{aligned} \quad (14)$$

where $\mathbf{a}_{i,1} = (\mathbf{x}_i \oplus \tilde{\mathbf{x}}_i)$, $\mathbf{a}_{i,2} = (x_i \oplus \tilde{x}_i) \mathbf{1}_{N-1 \times 1}$, $\mathbf{a}_{i,n} = \mathbf{G}_{i,n-2} \circ (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)$, for $3 \leq n \leq N+1$.

Using Lemma 3, (14) can be expressed as

$$\hat{\mathbf{x}}_{e,i} = \sum_{n=1}^{N+1} (-2)^{n-1} \sum_{1 \leq p_1 < p_2 < \cdots < p_n \leq N+1} \prod_{j=1}^n \circ \mathbf{a}_{i,p_j}. \quad (15)$$

Based on (15), using Bayesian formula, the error probability of user i can be calculated as

$$\begin{aligned} P_{e,i}(\mathbf{F}) &= E[|\hat{\mathbf{x}}_{e,i}|] \\ &= \sum_{k=1}^{N-1} E[(\hat{\mathbf{x}}_{e,i})_k] \\ &= \sum_{k=1}^{N-1} \sum_{n=1}^{N+1} (-2)^{n-1} \sum_{1 \leq p_1 < p_2 < \cdots < p_n \leq N+1} \prod_{j=1}^n E[(\mathbf{a}_{i,p_j})_k], \end{aligned} \quad (16)$$

where

$$\begin{aligned}
E[(\mathbf{a}_{i,1})_k] &= \begin{cases} E[x_k \oplus \tilde{x}_k], & k \leq i-1, \\ E[x_{k+1} \oplus \tilde{x}_{k+1}], & i \leq k \leq N-1, \end{cases} \\
E[(\mathbf{a}_{i,2})_k] &= E[x_i \oplus \tilde{x}_i], \\
E[(\mathbf{a}_{i,l})_k] &= E[r_k \oplus \tilde{r}_{i,k}] (\mathbf{F}_i^{-1})_{k,l-2}, \text{ for } 3 \leq l \leq N+1.
\end{aligned} \tag{17}$$

For BPSK, the relationship between the SEP and the received SNR over Rayleigh fading channels is [32]

$$P(e_\Delta) = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_\Delta}{1 + \gamma_\Delta}}, \tag{18}$$

where e_Δ denotes the error and γ_Δ denotes the average received SNR. Thus, (17) can be rewritten as

$$\begin{aligned}
E[(\mathbf{a}_{i,1})_k] &= \begin{cases} \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_{x_k}}{1 + \gamma_{x_k}}}, & k \leq i-1, \\ \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_{x_{k+1}}}{1 + \gamma_{x_{k+1}}}}, & i \leq k \leq N-1, \end{cases} \\
E[(\mathbf{a}_{i,2})_k] &= \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_{x_i}}{1 + \gamma_{x_i}}}, \\
E[(\mathbf{a}_{i,l})_k] &= \left(\frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_{i,r_k}}{1 + \gamma_{i,r_k}}} \right) (\mathbf{F}_i^{-1})_{k,l-2}, \text{ for } 3 \leq l \leq N+1,
\end{aligned} \tag{19}$$

where γ_{x_k} is the average received SNR of U_k at the relay and γ_{i,r_k} is the average received SNR at U_i in time slot k in the second phase.

Thus the error probability of the system can be expressed as

$$P_e(\mathbf{F}) = \frac{1}{N} \sum_{i=1}^N P_{e,i}(\mathbf{F}), \tag{20}$$

where $P_{e,i}(\mathbf{F})$ is given in (16).

The throughput of the system:

Here we define the throughput as the symbols received correctly at all the users per time slot.

Then the throughput of the system with NC is given by

$$\mathbb{T}_{NC} = \frac{N(N-1)(1-P_e(\mathbf{F}))}{2N-1}, \quad (21)$$

where N denotes the number of the users and $P_e(\mathbf{F})$ is given in (20).

As the extension of two-way relay network [33], for the system without NC, we first derive the SEP of the system. For one user, the error probability of other users receiving x_i can be expressed as

$$P_{e,x_i} = (N-1)P_e(x_i) + (1-P_e(x_i)) \sum_{j=1}^{N-1} P_e(r_{j,i}) \quad (22)$$

where $P_e(x_i)$ denotes the error probability of the relay detecting x_i and $P_e(r_{j,i})$ denotes the error probability of U_j detecting r_i . In the system without NC, it is evident that r_i is x_i .

Then the SEP of the system without NC can be expressed as

$$\begin{aligned} P_e' &= \sum_{i=1}^N \left\{ (N-1)P_e(x_i) + (1-P_e(x_i)) \sum_{j=1}^{N-1} P_e(r_{j,i}) \right\} \\ &= \sum_{i=1}^N \left\{ \frac{1}{2}(N-1) \left(1 - \sqrt{\frac{\gamma_{x_i}}{1+\gamma_{x_i}}} \right) + \frac{1}{2} \left(1 - \frac{1}{2} \left(1 - \sqrt{\frac{\gamma_{x_i}}{1+\gamma_{x_i}}} \right) \right) \sum_{j=1}^{N-1} \left(1 - \sqrt{\frac{\gamma_{j,r_i}}{1+\gamma_{j,r_i}}} \right) \right\} \end{aligned} \quad (23)$$

where Eq. (19) is used in the last step.

The throughput of the system without NC can be easily obtained as

$$\mathbb{T}_{No\ NC} = \frac{(N-1)(1-P_e')}{2}, \quad (24)$$

where P_e' is given in (23).

In the high SNR region, the throughput of the scheme with NC and without NC can be

expressed as

$$\begin{aligned}
\mathbb{T}_{NC}^{\infty} &= \lim_{SNR \rightarrow \infty} \mathbb{T}_{NC} \\
&= \lim_{Pe(\mathbf{F}) \rightarrow 0} \mathbb{T}_{NC} \\
&= \frac{N(N-1)}{2N-1}.
\end{aligned} \tag{25}$$

$$\begin{aligned}
\mathbb{T}_{No\ NC}^{\infty} &= \lim_{SNR \rightarrow \infty} \mathbb{T}_{No\ NC} \\
&= \frac{N-1}{2}.
\end{aligned} \tag{26}$$

where \mathbb{T}_{NC}^{∞} and $\mathbb{T}_{No\ NC}^{\infty}$ denote the throughput of the system with and without NC in the high SNR region, respectively.

Based on (25) (26), the absolute value of the throughput improvement equals to

$$\begin{aligned}
\mathbb{T}_{\Delta}^{\infty} &= \mathbb{T}_{NC}^{\infty} - \mathbb{T}_{No\ NC}^{\infty} \\
&= \frac{N(N-1)}{2N-1} - \frac{N-1}{2} \\
&= \frac{1}{4} \left(1 - \frac{1}{2N-1} \right).
\end{aligned} \tag{27}$$

It can be seen that $\mathbb{T}_{\Delta}^{\infty}$ increases with N , which indicates that increasing the number of the users brings an improved performance of the absolute value of the throughput.

C. System performance bound

Above, the exact closed-form expressions of SEP and throughput of the system have been derived. However, it can be seen that the expressions are very complex and provide few insights. In this subsection, the tight upper bound of SEP of the system will be derived to show useful insights.

Lemma 4: For addition and multiplication in GF(2), we have

$$a \oplus b \leq a + b, \tag{28}$$

$$(\mathbf{AB}) \bmod (2) \leq \mathbf{AB}, \quad (29)$$

where \mathbf{A} is a $(L \times M)$ matrix and \mathbf{B} is a column vector with M elements. Symbol a and b , all the elements in \mathbf{A} and \mathbf{B} are in $\text{GF}(2)$.

Proof: See Appendix D. ■

Using Lemma 4, (11) can be upper bounded as

$$\hat{\mathbf{x}}_{e,i} \leq (\mathbf{x}_i \oplus \tilde{\mathbf{x}}_i) + (\mathbf{1}_{N-1 \times 1} (x_i \oplus \tilde{x}_i)) + (\mathbf{F}_i^{-1} (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)) \triangleq \hat{\mathbf{x}}_{e,i}^U, \quad (30)$$

where in the high SNR region, the probability that two errors occur simultaneously is much lower than the probability that only one error occurs. Thus, the condition that more than two errors occur simultaneously can be ignored. We note that this simplified bound is still very tight as we can see from following simulations.

Based on (30), the error probability of user i can be calculated as

$$\begin{aligned} P_{e,i}(\mathbf{F}) &\leq E[|\hat{\mathbf{x}}_{e,i}^U|] \\ &= \sum_{k=1, k \neq i}^N E[x_k \oplus \tilde{x}_k] + (N-1) E[x_i \oplus \tilde{x}_i] + \sum_{k=1}^{N-1} E[r_k \oplus \tilde{r}_{i,k}] |\mathbf{G}_{i,k}| \triangleq P_{e,i}^U(\mathbf{F}), \end{aligned} \quad (31)$$

Using (31), the SEP of the system can be upper bounded as

$$P_e(\mathbf{F}) \leq \frac{1}{N} \sum_{i=1}^N P_{e,i}^U(\mathbf{F}) \triangleq P_e^U(\mathbf{F}). \quad (32)$$

Substituting (18) into (32), the final upper bound of the SEP of the system is

$$\begin{aligned} P_e^U(\mathbf{F}) &= \frac{1}{N} \sum_{i=1}^N \left\{ \sum_{k=1, k \neq i}^N \left(\frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_{x_k}}{1 + \gamma_{x_k}}} \right) \right. \\ &\quad \left. + (N-1) \left(\frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_{x_i}}{1 + \gamma_{x_i}}} \right) + \sum_{k=1}^{N-1} \left(\frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma_{i,r_k}}{1 + \gamma_{i,r_k}}} \right) |\mathbf{G}_{i,k}| \right\}. \end{aligned} \quad (33)$$

From the above expression (33), it can be seen that for U_i , the received error probability is made up of three parts: the first item $\sum_{k=1, k \neq i}^N P(e_{x_k})$ results from the transmissions of $N - 1$ users except U_i in the first phase, the second item $(N - 1) P(e_{x_i})$ results from the transmission of U_i itself in the first phase, the third item $\sum_{k=1}^{N-1} P(e_{i,r_k}) |\mathbf{G}_{i,k}|$ results from the transmissions of relay in the second phase. It is evident that the impacts of the three items on the SEP of the system are on the same order of magnitude, since their coefficients are all about $N - 1$. It can also be seen that the third item has the largest impact on the SEP of the system, since $|\mathbf{G}_{i,k}| \geq 1$. As only one relay is employed to assist the users and the users do not cooperative with each other in the system, the diversity order of the proposed scheme is 1.

D. Designed BMNC encoding matrix

Above, the connection between the system error performance and the network coding matrix is provided. Moreover, several design criteria of the network coding matrix, which ensure the successful information exchange, are also investigated. It can be seen from (33) that the network coding matrix has a significant impact on the system performance. Thus the network coding matrix should be designed carefully to further improve the system error performance.

In practical systems, the distance between the relay and the user varies for different users. Then the average received SNRs at the relay for different users may not be the same. Moreover, for high order modulations, the power that the relay uses also varies for different symbols.

It is assumed that the statistical channel state information is known at the relay. Without loss of generality, we assume that the statistic channel conditions between the users and the relay have an ascending order from U_1 to U_N , which means that the statistic distance between the U_i and the relay is larger than that between the U_j and the relay when $i < j$. Moreover, we assume that the power that the relay uses to broadcast the detected information has a descending order from time slot 1 to time slot $N - 1$, which means that the power that the relay uses in time slot i is higher than that in time slot j when $i < j$. In the following, we will show that the assumed order of the channel gains and that of the power allocation formulate the designed network coding matrix in a more detail.

To simplify the network coding design process and save the memory, we propose a puncturing operation based network coding matrix design scheme. In this scheme, the designed encoding matrix of N users is a matrix in which the upper left corner is the designed encoding matrix of $N - 1$ users. Thus, the relay only needs to memorize the designed network coding matrix of the maximum number users.

We design an encoding matrix for N users as

$$\mathbf{F}_{|N \text{ users}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}_{(N-1) \times N}, \quad (34)$$

the optimality of which will be discussed in Section V.

V. THE OPTIMALITY OF THE MATRIX GIVEN IN (34)

In this section, we will discuss the optimality of the matrix given in (34). First, we consider the situation that there are three users.

Lemma 5: For $N = 3$, the designed network coding matrix \mathbf{F} , which minimizes the bound of SEP, can be designed as follows

$$\mathbf{F}_{|3 \text{ users}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \quad (35)$$

Proof: See Appendix E. ■

In the following, the designed network coding matrices for the systems with more than three users will be discussed. We suppose the designed encoding matrix of $N - 1$ users is

a $(N - 2) \times (N - 1)$ matrix, which is evaluated by

$$\mathbf{F}_{|N-1 \text{ users}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}_{(N-2) \times (N-1)}. \quad (36)$$

Based on this assumption and Lemma 5, if the designed encoding matrix of N users is the matrix described in (34), using mathematical induction, the matrix given in (34) is the matrix we want. Using (36) and the proposed network coding matrix design scheme, the designed encoding matrix of N users can be written as

$$\mathbf{F}_{|N \text{ users}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & b_1 \\ 1 & 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & b_{N-2} \\ u_1 & u_2 & u_3 & \cdots & u_{N-1} & u_N \end{bmatrix}_{(N-1) \times N}, \quad (37)$$

where $b_1, b_2, \dots, b_{N-2}, u_1, u_2, \dots, u_N$ are unknown elements, which are in $\text{GF}(2)$.

Using Theorem 1, each element of the last column vector, b_1, b_2, \dots, b_{N-2} , should be the sum of the other elements in its row vector in $\text{GF}(2)$. For the first $N - 2$ row vectors, since there are just two “1” elements except the last column in one row, we have $b_1 = b_2 = \dots = b_{N-2} = 0$.

Since \mathbf{F}_i should be full rank for any i , each column vector of \mathbf{F}_i should not be a zero column vector. Thus, in the last column vector, u_N should be 1, since the other elements in this column vector are all 0. The designed encoding matrix of N users can be rewritten as

$$\mathbf{F}_{|N \text{ users}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ u_1 & u_2 & u_3 & \cdots & u_{N-1} & 1 \end{bmatrix}_{(N-1) \times N}, \quad (38)$$

where using Theorem 1, we have

$$\sum_{k=1}^{N-1} \oplus u_k = 1. \quad (39)$$

Thus, only u_1, u_2, \dots, u_{N-1} , which are the elements of the last row vector, are left to be designed to improve the system error performance. From (33), it can be seen that the network decoding matrices have considerable impact on the SEP of the system. In the following, we need to acquire the network decoding matrix \mathbf{F}_i^{-1} for $i \in [1, N]$. Elementary row operations are used to obtain the network decoding matrix.

For the first user, using elementary row operations, we have

$$\begin{aligned} [\mathbf{F}_{1|N \text{ users}} \quad \mathbf{I}] &= \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ u_2 & u_3 & \cdots & u_{N-1} & 1 & 0 & 0 & \cdots & 0 & 1 \end{array} \right]_{(N-1) \times (2N-2)} \\ &\rightarrow \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & u_2 & u_3 & \cdots & u_{N-1} & 1 \end{array} \right]_{(N-1) \times (2N-2)}, \end{aligned} \quad (40)$$

where in the last step, the first $N - 2$ row vectors are multiplied by different coefficient, for example the j th row vector is multiplied by u_{j+1} for $j \leq N - 2$, and then are all added to the

last row vector. From (40), the inverse matrix of \mathbf{F}_1 is

$$\mathbf{F}_{1|N \text{ users}}^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ u_2 & u_3 & \cdots & u_{N-1} & 1 \end{bmatrix}_{(N-1) \times (N-1)} = \mathbf{F}_1. \quad (41)$$

For the second user, using elementary row operations, we have

$$\begin{aligned} \left[\mathbf{F}_{2|N \text{ users}} \quad \mathbf{I} \right] &= \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ u_1 & u_3 & \cdots & u_{N-1} & 1 & 0 & 0 & \cdots & 0 & 1 \end{array} \right]_{(N-1) \times (2N-2)} \\ &\rightarrow \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & u_3 & \cdots & u_{N-1} & 1 & u_1 & 0 & \cdots & 0 & 1 \end{array} \right]_{(N-1) \times (2N-2)}, \end{aligned} \quad (42)$$

where in the last step, the first row vector is added directly to other $N-3$ row vectors except the last row vector, which is added by the first row vector multiplied by u_1 . Then for $2 \leq j \leq N-2$, multiplying the j th row vector with u_{j+1} and adding the product to the last row vector, (42) can

be rewritten as

$$\left[\mathbf{F}_{2|N \text{ users}} \quad \mathbf{I} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{array} \middle| \begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \\ \sum_{k=1, k \neq 2}^{N-1} \oplus u_k & u_3 & \cdots & u_{N-1} & 1 \end{array} \right]_{(N-1) \times (2N-2)} . \quad (43)$$

From (43), the inverse matrix of \mathbf{F}_2 can be expressed as

$$\mathbf{F}_{2|N \text{ users}}^{-1} = \left[\begin{array}{ccccc} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \\ \sum_{k=1, k \neq 2}^{N-1} \oplus u_k & u_3 & \cdots & u_{N-1} & 1 \end{array} \right]_{(N-1) \times (N-1)} . \quad (44)$$

In the same way as the inverse matrix of \mathbf{F}_1 and \mathbf{F}_2 , the inverse matrix of \mathbf{F}_i for $i \in [3, N-1]$ can be described as

$$\mathbf{F}_{i|N \text{ users}}^{-1} = \left[\begin{array}{cccc|ccccc} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 1 & 0 \\ u_2 & u_3 & \cdots & u_{i-1} & \sum_{k=1, k \neq i}^{N-1} \oplus u_k & u_{i+1} & \cdots & u_{N-1} & 1 \end{array} \right]_{(N-1) \times (N-1)} , \quad (45)$$

and the inverse matrix of \mathbf{F}_N equals to

$$\mathbf{F}_{N|N \text{ users}}^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ u_2 & u_3 & \cdots & u_{N-1} & 1 \end{bmatrix}_{(N-1) \times (N-1)}. \quad (46)$$

We design a matrix $\widehat{\mathbf{F}}_{|N \text{ users}}$, which can be shown in (34). In the following, we will prove that using $\widehat{\mathbf{F}}_{|N \text{ users}}$ as the network coding matrix, the upper bound of the SEP of the system is minimized.

Based on (41), (44), (45) and (46), using (32), we have

$$\begin{aligned} & N \left(P_e(\mathbf{F}_{|N \text{ users}}) - P_e(\widehat{\mathbf{F}}_{|N \text{ users}}) \right) \\ &= \sum_{k=1}^{N-2} E[r_k \oplus \tilde{r}_{1,k}] u_{k+1} + E[r_1 \oplus \tilde{r}_{2,1}] \left(\sum_{k=1, k \neq 2}^{N-1} \oplus u_k - 1 \right) + \sum_{k=2}^{N-2} E[r_k \oplus \tilde{r}_{2,k}] u_{k+1} \\ &+ \sum_{i=3}^{N-1} \left\{ \sum_{k=1, k \neq i-1}^{N-2} E[r_k \oplus \tilde{r}_{i,k}] u_{k+1} + E[r_{i-1} \oplus \tilde{r}_{i,i-1}] \left(\sum_{k=1, k \neq i}^{N-1} \oplus u_k - 1 \right) \right\} \\ &+ \sum_{k=1}^{N-2} E[r_k \oplus \tilde{r}_{N,k}] u_{k+1}. \end{aligned} \quad (47)$$

Using (39), we have $\sum_{k=1, k \neq i}^{N-1} \oplus u_k - 1 = -u_i$ and (47) can be expressed as

$$\begin{aligned}
& N \left(P_e(\mathbf{F}_{|N \text{ users}}) - P_e(\widehat{\mathbf{F}}_{|N \text{ users}}) \right) \\
&= \sum_{k=1}^{N-2} E[r_k \oplus \tilde{r}_{i,k}] u_{k+1} - E[r_1 \oplus \tilde{r}_{2,1}] u_2 + \sum_{k=2}^{N-2} E[r_k \oplus \tilde{r}_{2,k}] u_{k+1} \\
&\quad + \sum_{i=3}^{N-1} \left\{ \sum_{k=1, k \neq i-1}^{N-2} E[r_k \oplus \tilde{r}_{i,k}] u_{k+1} - E[r_{i-1} \oplus \tilde{r}_{i,i-1}] u_i \right\} + \sum_{k=1}^{N-2} E[r_k \oplus \tilde{r}_{N,k}] u_{k+1} \\
&= \sum_{k=1}^{N-2} (E[r_k \oplus \tilde{r}_{1,k}] - E[r_k \oplus \tilde{r}_{k+1,k}]) u_{k+1} + \sum_{k=2}^{N-2} E[r_k \oplus \tilde{r}_{2,k}] u_{k+1} \\
&\quad + \sum_{i=3}^{N-1} \sum_{k=1, k \neq i-1}^{N-2} E[r_k \oplus \tilde{r}_{i,k}] u_{k+1} + \sum_{k=1}^{N-2} E[r_k \oplus \tilde{r}_{N,k}] u_{k+1} \\
&\geq 0,
\end{aligned} \tag{48}$$

where $E[r_k \oplus \tilde{r}_{1,k}] > E[r_k \oplus \tilde{r}_{k+1,k}]$ for $k \in [1, N-2]$ is used in the last inequality. From the above expression, since the coefficients of u_k 's are all strictly positive, it can be seen that the requirement of $N \left(P_e(\mathbf{F}_{|N \text{ users}}) - P_e(\widehat{\mathbf{F}}_{|N \text{ users}}) \right) = 0$ is that $u_k = 0, k = 2, 3, \dots, N-1$, which indicates that only $\widehat{\mathbf{F}}_{|N \text{ users}}$ meets the requirement. Thus the proposed encoding matrix for N users is unique.

Thus $\widehat{\mathbf{F}}$ is the designed encoding matrix for N users, which minimizes the upper bound of the SEP of the system. Based on Lemma 5, using mathematical induction, we have the following theorem.

Theorem 6: The designed network coding matrix of N users, which minimizes the bound of SEP, is given in (34).

From Theorem 6, it can be seen that the designed network coding matrix is structured and sparse. The properties of the matrix simplify the encoding and decoding process while improving system performance.

In practical systems, the relay first needs to know the number of the users. Moreover, the statistical information of the user to relay channels should be available to the relay. After receiving the necessary information, the designed NC matrix is constructed based on its closed-form

expression given by Theorem 6. According to the designed NC matrix, the users will send their information to the relay in turn. The relay will detect, encode, and broadcast the received information. Finally, each user decodes its received information by exploiting its own information.

VI. SIMULATION RESULTS

In this section, the performance of the analytical results will be compared with Monte Carlo simulations.

As discussed in the Section IV-C, in practical systems, the received power at the relay for different users is different and so does the transmit power of the relay at different time slot. The conditions of the simulations are set as follows: the transmit power at different users is the same and the average received SNR of U_i at the relay is 3dB worse than that of U_{i+1} , due to the different distances between the users and the relay; the power that the relay uses in time slot i is 3dB higher than that in time slot $i + 1$, since the power gap between two adjacent bits is 3dB in some high order modulations; the smallest transmit power at the relay and the transmit power at one user are the same; the smallest received SNR at the U_1 in the second phase of the system with NC is denoted as E_s/N_0 in the following figures; BPSK modulation is considered. We assume the same total transmit power of the system with and without NC.

Fig. 2 presents the throughput performance of the system with and without network coding, for 4, 5, and 6 users. The “X users with NC” curves are generated by combining (20) and (21), and the “X users without NC” curves are generated by combining (23) and (24). The NC matrices are given by (34). From the figure, it can be seen that compared to the system without NC, NC improves the throughput about 0.21, 0.22, 0.23 for 4, 5, 6 users respectively. This is predicted by (27), which indicates that as the number of the users increases, the absolute value of the throughput improvement increases. Moreover, for 4 users, it can be seen that network coding improves the throughput about 14% in the high SNR region, while the improvement is about 11% for 5 users and about 9% for 6 users. For a practical communication service, such as video conference in which each user may want to have a discussion with the other users, the number of users is usually limited, e.g., 3 or 4 people when using DamakaTM [34]. Furthermore,

it is natural that the error performance gain will decrease as the number of users increases for any MAC strategy. This indicates that our proposed scheme can improve the throughput of the system much in widely used scenarios.

In Fig. 3, the simulated and tight upper bounds of the SEP performance of the system with network coding are compared for different number of users. The “numerical” curves are generated by (33) and the NC matrices are given by (34). We can see that the numerical SEP curves accurately predict the simulation ones. From the figure, the SEP performance is slightly higher as the number of users increases. It means that the interference between the users is small, which is caused by the network coding at the relay. Thus when the number of the users is large, NC is still efficient in improving the throughput with not so much impact on the system error performance.

In Fig. 4, the SEP performance of the system is compared with different network coding matrix, for 4 users, where the matrices 1, 2, 3 are, respectively,

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It can be seen that our designed matrix can improve the SEP performance compared to an ad-hoc coding matrix. However, compared to the system without NC, our proposed scheme achieves a slightly poor SEP performance. The above simulations assume the same channel conditions of the source to relay channel and the corresponding relay to source channel. However, in Fig. 5, we will show that our proposed scheme may achieve a better SEP performance in other channel conditions.

In Fig. 5 when the source to relay link is 20dB better than the corresponding relay to source link, it can be seen that our proposed scheme achieves about 2dB SEP gain compared to the system without NC. This indicates that our proposed scheme can improve both the throughput and the error performance in the system where the source to relay link is better than the corresponding relay to source link, such as one satellite assists the information exchange of several base stations.

VII. CONCLUSIONS

We have investigated the design of binary linear NC for N -way relay networks, where N end nodes exchange their information with the assistance of one N -way relay. NC matrix in GF(2) was proposed to describe the linear NC process. The design criteria of the NC matrix, which improve the SEP performance, were provided. Moreover, the closed form expressions and the upper bound of SEP of the system were given. It can be seen that using linear NC, the throughput gain of the system is more than 10% for less than 6 users. To improve the system performance further, we designed NC matrices for arbitrary number of users, which minimized the bound of SEP.

APPENDIX A

PROOF OF THEOREM 1

First, we need to prove that it is the necessary condition. We suppose that all the network sub-encoding matrices are full rank. Since \mathbf{F}_i is full rank, the $N-1$ column vectors of \mathbf{F}_i form a $(N-1)$ -dimensional linear vector space in GF(2) and any other column vector in GF(2) which contains $N-1$ elements is in this linear vector space. Then \mathbf{f}_i can be expressed as a kind of linear combining of the column vectors of \mathbf{F}_i . That is

$$\mathbf{f}_i = \sum_{k=1, k \neq i}^N \oplus \beta_k \mathbf{f}_k, \quad (49)$$

where β_k can only be 0 or 1. For any user j , $j \neq i$, if β_j in (49) is 0, then \mathbf{F}_j only has no more than $N-2$ linearly independent column vectors. Thus, \mathbf{F}_j is not full rank. That is a contradiction of the hypothesis. So β_k in (49) should be 1, for $j \in [1, N]$, $j \neq i$. Thus we have proved that it is the necessary condition.

Second, we need to prove that it is the sufficient condition. For $j, j \neq i$, \mathbf{F}_j can be spread as

$$\begin{aligned} \mathbf{F}_j &= \begin{bmatrix} \mathbf{f}_1 & \cdots & \mathbf{f}_{i-1} & \mathbf{f}_i & \mathbf{f}_{i+1} & \cdots & \mathbf{f}_{j-1} & \mathbf{f}_{j+1} & \cdots & \mathbf{f}_N \end{bmatrix}_{(N-1) \times (N-1)} \\ &= \begin{bmatrix} \mathbf{f}_1 & \cdots & \mathbf{f}_{i-1} & \sum_{k=1, k \neq i}^N \oplus \mathbf{f}_k & \mathbf{f}_{i+1} & \cdots & \mathbf{f}_{j-1} & \mathbf{f}_{j+1} & \cdots & \mathbf{f}_N \end{bmatrix}_{(N-1) \times (N-1)}, \end{aligned} \quad (50)$$

where (10) is used in the last step. Then we add all the column vectors except the i th column vector onto the i th column in GF(2), which is the elementary column operation on \mathbf{F}_j . Thus \mathbf{F}_j can be expressed as

$$\begin{aligned}\mathbf{F}_j &\rightarrow \begin{bmatrix} \mathbf{f}_1 & \cdots & \mathbf{f}_{i-1} & \mathbf{f}_j & \mathbf{f}_{i+1} & \cdots & \mathbf{f}_{j-1} & \mathbf{f}_{j+1} & \cdots & \mathbf{f}_N \end{bmatrix}_{(N-1) \times (N-1)} \\ &\rightarrow \begin{bmatrix} \mathbf{f}_1 & \cdots & \mathbf{f}_{i-1} & \mathbf{f}_{i+1} & \cdots & \mathbf{f}_{j-1} & \mathbf{f}_j & \mathbf{f}_{j+1} & \cdots & \mathbf{f}_N \end{bmatrix}_{(N-1) \times (N-1)} \\ &= \mathbf{F}_i.\end{aligned}\tag{51}$$

From (51), it can be seen that \mathbf{F}_j has the same rank of \mathbf{F}_i , which means that \mathbf{F}_j is full rank if \mathbf{F}_i is full rank. Thus we have proved that it is the sufficient condition and the theorem is proved.

APPENDIX B

PROOF OF THEOREM 2

Using (2), Eq. (9) can be rewritten as

$$\begin{aligned}\hat{\mathbf{x}}_i &= \{\mathbf{F}_i^{-1} \{\mathbf{r} \oplus \mathbf{r} \oplus \tilde{\mathbf{r}}_i \oplus \{(\mathbf{f}_i x_i) \bmod (2)\}\}\} \bmod (2) \\ &= \{\mathbf{F}_i^{-1} (\mathbf{r} \oplus \{(\mathbf{f}_i x_i) \bmod (2)\} \oplus (\mathbf{r} \oplus \tilde{\mathbf{r}}_i))\} \bmod (2) \\ &= \left\{ \mathbf{F}_i^{-1} \left(\underbrace{\{(\mathbf{F}\tilde{\mathbf{x}}) \bmod (2)\} \oplus \{(\mathbf{f}_i x_i) \bmod (2)\}}_{\triangleq I_1} \oplus (\mathbf{r} \oplus \tilde{\mathbf{r}}_i) \right) \right\} \bmod (2).\end{aligned}\tag{52}$$

The item I_1 in the above equation can be expressed as

$$I_1 = \begin{bmatrix} \sum_{k=1}^{k=N} \oplus f_{1,k} \tilde{x}_k \\ \sum_{k=1}^{k=N} \oplus f_{2,k} \tilde{x}_k \\ \cdots \\ \sum_{k=1}^{k=N} \oplus f_{N-1,k} \tilde{x}_k \end{bmatrix}_{(N-1) \times 1} \oplus \begin{bmatrix} f_{1,i} x_i \\ f_{2,i} x_i \\ \cdots \\ f_{N-1,i} x_i \end{bmatrix}_{(N-1) \times 1},\tag{53}$$

and the above expression can be rewritten as

$$\begin{aligned}
I_1 &= \left[\begin{array}{c} \sum_{k=1, k \neq i}^{k=N} \oplus f_{1,k} \tilde{x}_k \\ \sum_{k=1, k \neq i}^{k=N} \oplus f_{2,k} \tilde{x}_k \\ \dots \\ \sum_{k=1, k \neq i}^{k=N} \oplus f_{N-1,k} \tilde{x}_k \end{array} \right]_{(N-1) \times 1} \oplus \left[\begin{array}{c} f_{r_1, x_i}(x_i \oplus \tilde{x}_i) \\ f_{r_2, x_i}(x_i \oplus \tilde{x}_i) \\ \dots \\ f_{r_{N-1}, x_i}(x_i \oplus \tilde{x}_i) \end{array} \right]_{(N-1) \times 1} \\
&= \{(\mathbf{F}_i \tilde{\mathbf{x}}_i) \bmod (2)\} \oplus \{(\mathbf{f}_i(x_i \oplus \tilde{x}_i)) \bmod (2)\},
\end{aligned} \tag{54}$$

substituting (54) into (52), (52) can be written as

$$\hat{\mathbf{x}}_i = \{\mathbf{F}_i^{-1} \{(\mathbf{F}_i \tilde{\mathbf{x}}_i) \bmod (2)\} \oplus \{(\mathbf{f}_i(x_i \oplus \tilde{x}_i)) \bmod (2)\} \oplus (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)\} \bmod (2). \tag{55}$$

Since GF(2) is Galois field, distributive law of multiplication exists. Using distributive law of multiplication, (55) equals to

$$\begin{aligned}
\hat{\mathbf{x}}_i &= \{(\mathbf{F}_i^{-1} \mathbf{F}_i \tilde{\mathbf{x}}_i) \bmod (2)\} \oplus \{(\mathbf{F}_i^{-1} \mathbf{f}_i(x_i \oplus \tilde{x}_i)) \bmod (2)\} \oplus \{(\mathbf{F}_i^{-1} (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)) \bmod (2)\} \\
&= \tilde{\mathbf{x}}_i \oplus \{(\mathbf{F}_i^{-1} \mathbf{f}_i(x_i \oplus \tilde{x}_i)) \bmod (2)\} \oplus \{(\mathbf{F}_i^{-1} (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)) \bmod (2)\}.
\end{aligned} \tag{56}$$

Using Theorem 1, since \mathbf{F}_i should be full rank, we have

$$\begin{aligned}
\mathbf{f}_i &= \sum_{k=1, k \neq i}^N \oplus \mathbf{f}_k \\
&= \left(\left[\begin{array}{cccccc} \mathbf{f}_1 & \dots & \mathbf{f}_{i-1} & \mathbf{f}_{i+1} & \dots & \mathbf{f}_N \end{array} \right] \mathbf{1}_{N-1 \times 1} \right) \bmod (2) \\
&= (\mathbf{F}_i \mathbf{1}_{N-1 \times 1}) \bmod (2).
\end{aligned} \tag{57}$$

Based on (57), Eq. (56) can be expressed as

$$\hat{\mathbf{x}}_i = \tilde{\mathbf{x}}_i \oplus ((x_i \oplus \tilde{x}_i) \mathbf{1}_{N-1 \times 1}) \oplus \{(\mathbf{F}_i^{-1} (\mathbf{r} \oplus \tilde{\mathbf{r}}_i)) \bmod (2)\}, \tag{58}$$

where $\mathbf{F}_i^{-1} \mathbf{F}_i = \mathbf{I}$ is used. Thus the theorem is proved.

APPENDIX C

PROOF OF LEMMA 3

Mathematical induction is proposed to prove the theorem. First, consider the addition of two numbers in GF(2),

$$a \oplus b = a + b - 2ab. \quad (59)$$

Then we assume that the addition of Q numbers in GF(2) can be written as

$$\sum_{q=1}^Q \oplus a_q = \sum_{q=1}^Q (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \dots < p_q \leq Q} \prod_{j=1}^q a_{p_j}. \quad (60)$$

Using (59), the addition of $Q + 1$ numbers in GF(2) can be expressed as

$$\begin{aligned} \sum_{q=1}^{Q+1} \oplus a_q &= a_{Q+1} \oplus \sum_{q=1}^Q \oplus a_q \\ &= a_{Q+1} + \sum_{q=1}^Q \oplus a_q - 2a_{Q+1} \sum_{q=1}^Q \oplus a_q, \end{aligned} \quad (61)$$

and taking (60) into the above expression, (61) can be rewritten as

$$\begin{aligned} \sum_{q=1}^{Q+1} \oplus a_q &= a_{Q+1} + \sum_{q=1}^Q (-2)^{q-1} \sum_{q=1}^Q (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \dots < p_q \leq Q} \prod_{j=1}^q a_{p_j} \\ &\quad - 2a_{Q+1} \sum_{q=1}^Q (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \dots < p_q \leq Q} \prod_{j=1}^q a_{p_j}. \end{aligned} \quad (62)$$

Separate the $q = 1$ term from the second item of (62) and separate the $q = Q + 1$ term from the last item of (62), we have

$$\begin{aligned} \sum_{q=1}^{Q+1} \oplus a_q &= a_{Q+1} + \sum_{p_1}^Q a_{p_1} + \sum_{q=2}^Q (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \dots < p_q \leq Q} \prod_{j=1}^q a_{p_j} \\ &\quad + \sum_{q=2}^Q (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \dots < p_q \leq Q+1} \prod_{j=1}^q a_{p_j} + (-2)^Q \prod_{p_1=1}^{Q+1} a_{p_1}. \end{aligned} \quad (63)$$

Next, combining the first item and second item of (64), and combining the third item and

forth item of (64), the above expression can be reformulated as

$$\begin{aligned}
\sum_{q=1}^{Q+1} \oplus a_q &= \sum_{p_1}^{Q+1} a_{p_1} + \sum_{q=2}^{Q+1} (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \dots < p_q \leq Q+1} \prod_{j=1}^q a_{p_j} + (-2)^Q \prod_{p_1=1}^{Q+1} a_{p_1} \\
&= \sum_{q=1}^{Q+1} (-2)^{q-1} \sum_{1 \leq p_1 < p_2 < \dots < p_q \leq Q+1} \prod_{j=1}^q a_{p_j}.
\end{aligned} \tag{64}$$

From the above derivation, (12) is convenient for the addition of two numbers in GF(2). Moreover, based on the addition of Q numbers in GF(2), (12) is also convenient for the addition of $Q + 1$ numbers in GF(2). Thus, using mathematical induction, for arbitrary numbers, the addition in GF(2) can be expressed as (12) and Lemma 3 is proved.

APPENDIX D

PROOF OF LEMMA 4

Eq. (28) is obvious and we focus on (29). We define that $[\mathbf{A}]_{i,j} = a_{i,j}$ and $[\mathbf{B}]_i = b_i$. Then the multiplication between matrix \mathbf{A} and \mathbf{B} in GF(2) can be expressed as

$$(\mathbf{AB}) \bmod (2) = \begin{bmatrix} \sum_{i=1}^M \oplus a_{1,i} b_i \\ \sum_{i=1}^M \oplus a_{2,i} b_i \\ \vdots \\ \sum_{i=1}^M \oplus a_{L,i} b_i \end{bmatrix}_{M \times 1} \leq \begin{bmatrix} \sum_{i=1}^M a_{1,i} b_i \\ \sum_{i=1}^M a_{2,i} b_i \\ \vdots \\ \sum_{i=1}^M a_{L,i} b_i \end{bmatrix}_{M \times 1} = \mathbf{AB}. \tag{65}$$

where $a \oplus b \leq a + b$ is used in the inequality.

Thus we have (29) and the lemma is proved.

APPENDIX E

PROOF OF LEMMA 5

For $N = 3$, from (32), the error probability of the system in this situation can be calculated as

$$\begin{aligned}
3P_e &\leq 4(E[x_1 \oplus \tilde{x}_1] + E[x_2 \oplus \tilde{x}_2] + E[x_3 \oplus \tilde{x}_3]) \\
&\quad + (f_{2,2} + f_{2,3}) E[r_1 \oplus \tilde{r}_{1,1}] + (f_{1,2} + f_{1,3}) E[r_2 \oplus \tilde{r}_{1,2}] \\
&\quad + (f_{2,1} + f_{2,3}) E[r_1 \oplus \tilde{r}_{2,1}] + (f_{1,1} + f_{1,3}) E[r_2 \oplus \tilde{r}_{2,2}] \\
&\quad + (f_{2,1} + f_{2,2}) E[r_1 \oplus \tilde{r}_{3,1}] + (f_{1,1} + f_{1,2}) E[r_2 \oplus \tilde{r}_{3,2}] \\
&= 4((x_1 \oplus \tilde{x}_1) + (x_2 \oplus \tilde{x}_2) + (x_3 \oplus \tilde{x}_3)) \\
&\quad + (E[r_2 \oplus \tilde{r}_{2,2}] + E[r_2 \oplus \tilde{r}_{3,2}]) f_{1,1} + (E[r_1 \oplus \tilde{r}_{2,1}] + E[r_1 \oplus \tilde{r}_{3,1}]) f_{2,1} \\
&\quad + (E[r_2 \oplus \tilde{r}_{1,2}] + E[r_2 \oplus \tilde{r}_{3,2}]) f_{1,2} + (E[r_1 \oplus \tilde{r}_{1,1}] + E[r_1 \oplus \tilde{r}_{3,1}]) f_{2,2} \\
&\quad + (E[r_2 \oplus \tilde{r}_{1,2}] + E[r_2 \oplus \tilde{r}_{2,2}]) f_{1,3} + (E[r_1 \oplus \tilde{r}_{1,1}] + E[r_1 \oplus \tilde{r}_{2,1}]) f_{2,3}.
\end{aligned} \tag{66}$$

To ensure that each user can obtain other two users' information, the column vector of the encoding matrix \mathbf{F} can only be $[1, 1]^T$, $[0, 1]^T$ and $[1, 0]^T$. Using the assumptions of the statistic channel conditions between the users and the relay and the power that the relay uses to broadcast the detected information, we have

$$\begin{aligned}
E[r_1 \oplus \tilde{r}_{1,1}] &> E[r_1 \oplus \tilde{r}_{2,1}] > E[r_1 \oplus \tilde{r}_{3,1}], \\
E[r_2 \oplus \tilde{r}_{1,2}] &> E[r_2 \oplus \tilde{r}_{2,2}] > E[r_2 \oplus \tilde{r}_{3,2}], \\
E[r_2 \oplus \tilde{r}_{1,2}] &> E[r_1 \oplus \tilde{r}_{1,1}], \\
E[r_2 \oplus \tilde{r}_{2,2}] &> E[r_1 \oplus \tilde{r}_{2,1}], \\
E[r_2 \oplus \tilde{r}_{3,2}] &> E[r_1 \oplus \tilde{r}_{3,1}].
\end{aligned} \tag{67}$$

From the above relationship, it can be seen that

$$\begin{aligned}
& E[r_1 \oplus \tilde{r}_{2,1}] + E[r_1 \oplus \tilde{r}_{3,1}] \\
& < \min(E[r_2 \oplus \tilde{r}_{2,2}] + E[r_2 \oplus \tilde{r}_{3,2}], E[r_1 \oplus \tilde{r}_{1,1}] + E[r_1 \oplus \tilde{r}_{3,1}]), \\
& \max(E[r_2 \oplus \tilde{r}_{2,2}] + E[r_2 \oplus \tilde{r}_{3,2}], E[r_1 \oplus \tilde{r}_{1,1}] + E[r_1 \oplus \tilde{r}_{3,1}]), \\
& < \min(E[r_2 \oplus \tilde{r}_{1,2}] + E[r_2 \oplus \tilde{r}_{3,2}], E[r_2 \oplus \tilde{r}_{1,2}] + E[r_2 \oplus \tilde{r}_{3,2}], E[r_1 \oplus \tilde{r}_{1,1}] + E[r_1 \oplus \tilde{r}_{2,1}]).
\end{aligned} \tag{68}$$

Thus, $E[r_1 \oplus \tilde{r}_{2,1}] + E[r_1 \oplus \tilde{r}_{3,1}]$ is the smallest one and to minimize the SEP of the system, in (66), the coefficient of $E[r_1 \oplus \tilde{r}_{2,1}] + E[r_1 \oplus \tilde{r}_{3,1}]$ should be largest and then we have $f_{2,1} = 1$. Moreover, $E[r_2 \oplus \tilde{r}_{2,2}] + E[r_2 \oplus \tilde{r}_{3,2}]$ and $E[r_1 \oplus \tilde{r}_{1,1}] + E[r_1 \oplus \tilde{r}_{3,1}]$ are smaller than the rest, which result in $f_{1,1} = f_{2,2} = 1$. Thus (35) is proved.

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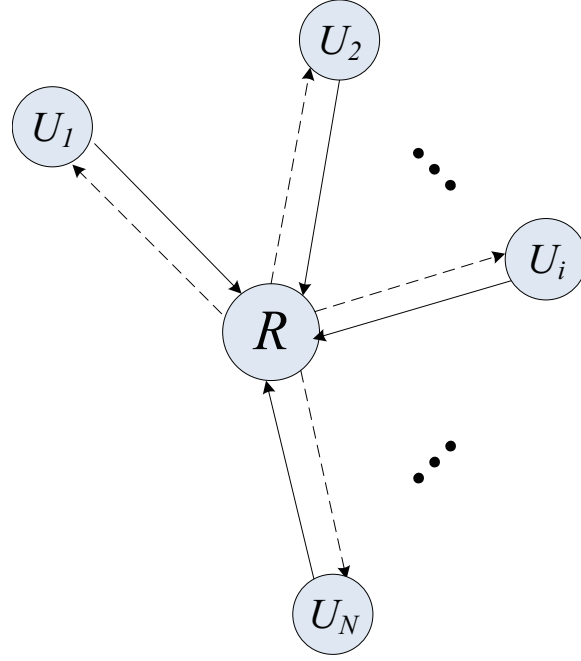


Fig. 1. System model.

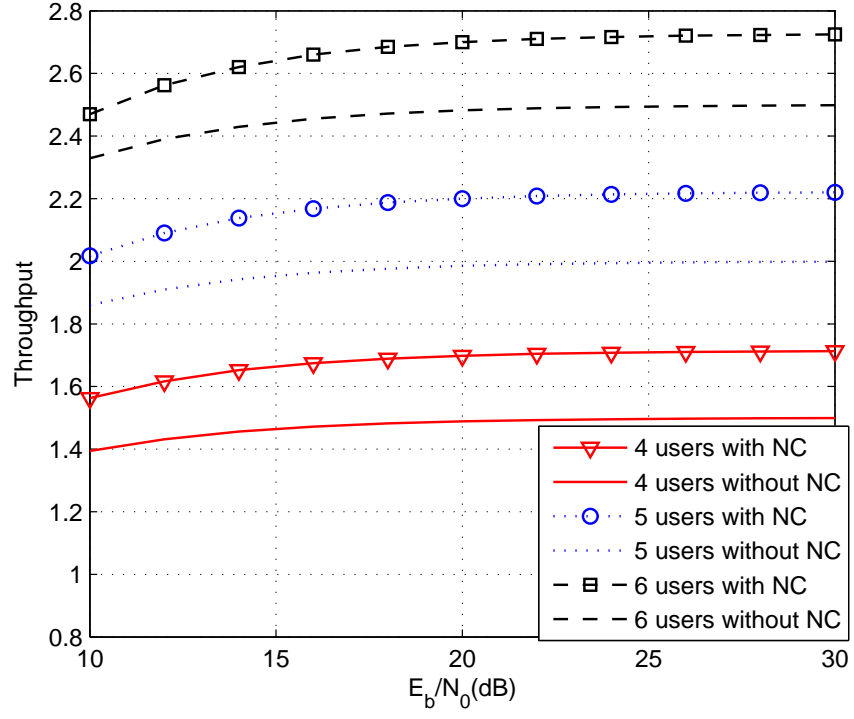


Fig. 2. Comparisons between the throughput of the system with and without network coding, for 4, 5, and 6 users.

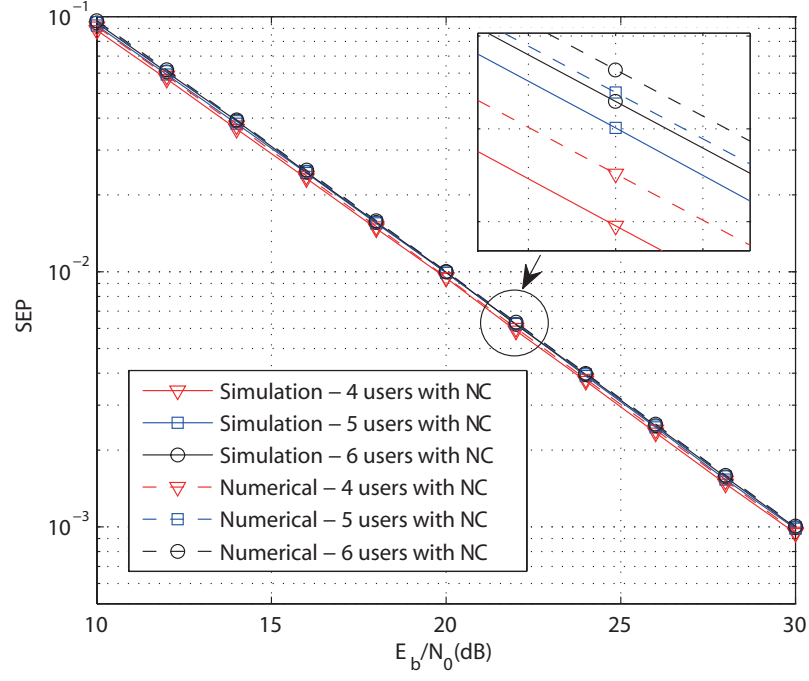


Fig. 3. The SEP of the system with network coding, for 4, 5, and 6 users.

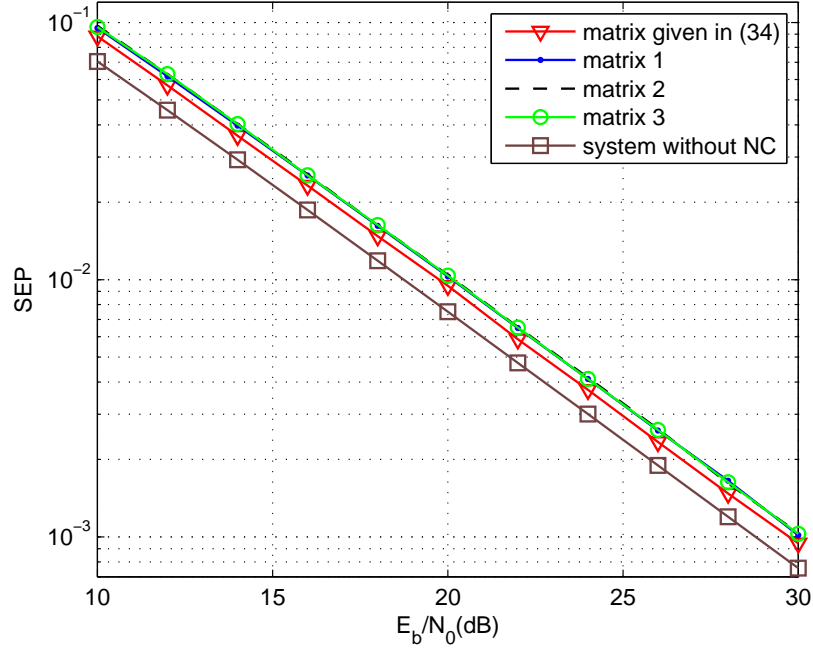


Fig. 4. The SEP of the system with different network coding matrix, for 4 users.

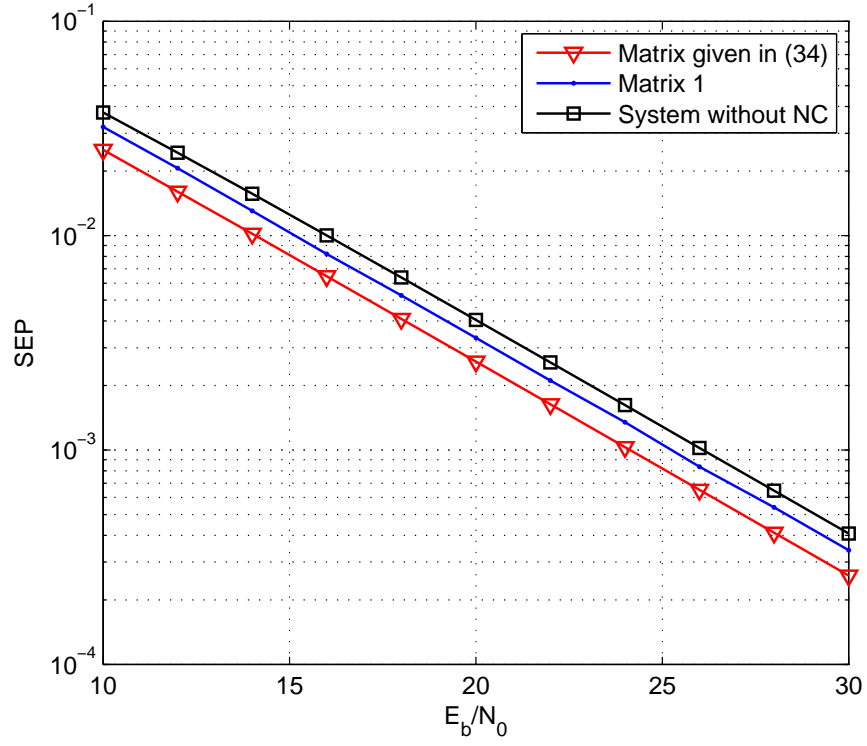


Fig. 5. The SEP of the system with different network coding matrix when the source to relay link is 20dB better than the corresponding relay to source link, for 4 users.